Definitions and key facts for section 1.9

The $n \times n$ **identity matrix** I_n is the $n \times n$ square matrix with 1 in every diagonal entry and 0s elsewhere.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad I_5 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ denote the columns of I_n.

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ in } \mathbb{R}^2 \quad e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^3 \quad e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ in } \mathbb{R}^5$$

The **standard matrix for the linear transformation** $T : R^n \to R^m$ is the $m \times n$ matrix

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

Fact: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and A is its standard matrix, then A is unique and

$$T(\mathbf{x}) = A\mathbf{x} \quad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **onto** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at *least* one $\mathbf{x}$ in $\mathbb{R}^n$.

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be **one-to-one** $\mathbb{R}^m$ if each $\mathbf{b}$ in $\mathbb{R}^m$ is the image of at *most* one $\mathbf{x}$ in $\mathbb{R}^n$.

Fact: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A, then

1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$.
2. T is one-to-one if and only if the columns of A are linearly independent.